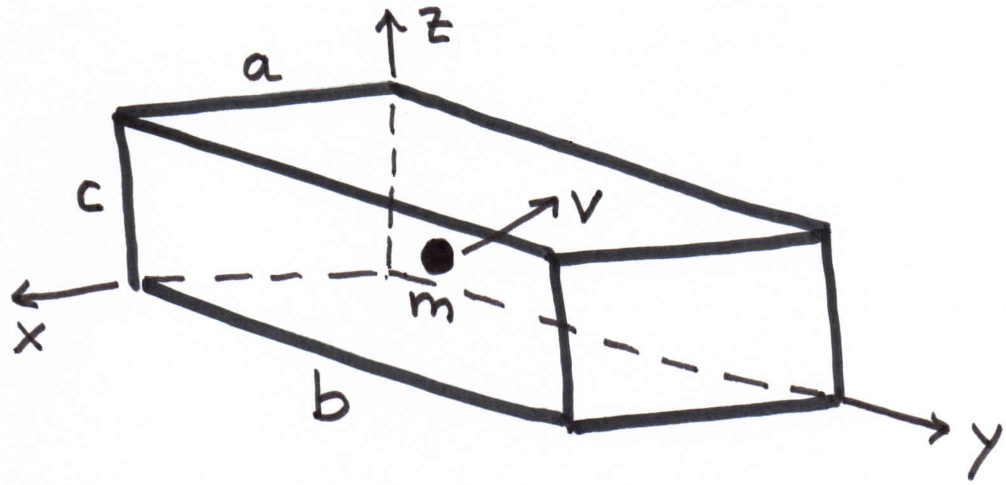
Particle in 3-D Box

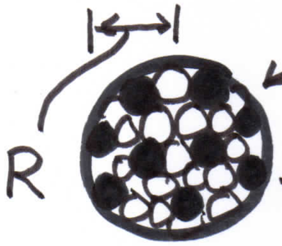


$$\psi_{n_x n_y n_z}(x,y,z) =$$

$$\sqrt{\frac{8}{abc}}\,\sin\left(\frac{\pi n_x x}{a}\right)\sin\left(\frac{\pi n_y y}{b}\right)\sin\left(\frac{\pi n_z z}{c}\right)$$

$$E_{n_x n_y n_z} = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)$$

Electron Contained in Nucleus?



Z protons
N neutrons

$\Big\}$ $A = Z + N$ nucleons

$1\,\text{fm} = 1\times10^{-15}\,\text{m}$

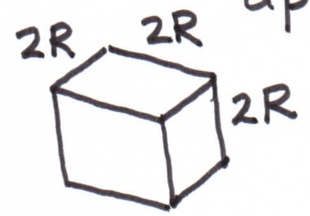
$$R \approx r_0 A^{1/3} \qquad r_0 = 1.3\ \text{fm}$$

example: carbon-12 (^{12}C)

$$R \approx 1.3\,\text{fm}\,(12)^{1/3} = 3.0\times10^{0}\,\text{fm} = 3\times10^{-6}\,\text{nm}$$

E_{111} = lowest energy

approx.



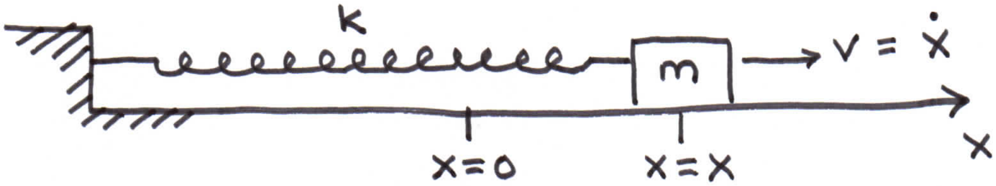
$$= \frac{h^2}{8m}\left(\frac{1^2}{a^2}+\frac{1^2}{b^2}+\frac{1^2}{c^2}\right)$$

$$= \frac{(hc)^2}{8mc^2}\left(\frac{1}{(6\times10^{-6})^2}+\frac{1}{(6\times10^{-6})^2}+\frac{1}{(6\times10^{-6})^2}\right)$$

$$= \frac{1240^2}{8(511{,}000)}\left[\frac{3}{(6\times10^{-6})^2}\right]$$

$$= 3.1\times10^{10}\,\text{eV} = \boxed{31\ \text{GeV}}$$

Classical S.H.O.



$$\vec{F} = -k\vec{x} \rightarrow \quad m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0 \leftarrow \text{second-order O.D.E.}$$

Solution is: $x(t) = c_1 \sin\sqrt{\frac{k}{m}}\,t + c_2 \cos\sqrt{\frac{k}{m}}\,t$

where the constants c_1 and c_2 are determined by initial conditions:

$$x(t=0) = c_1 \sin\sqrt{\frac{k}{m}}(0) + c_2 \cos\sqrt{\frac{k}{m}}(0) = c_2 \text{ (call } x_0\text{)}$$

$$\dot{x}(t=0) = c_1\sqrt{\frac{k}{m}} \cos\sqrt{\frac{k}{m}}(0) - c_2\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}(0) = c_1\sqrt{\frac{k}{m}}$$

call v_0

So $\quad x(t) = v_0\sqrt{\frac{m}{k}} \sin\sqrt{\frac{k}{m}}\,t + x_0 \cos\sqrt{\frac{k}{m}}\,t$

Consider the energy equation:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

amplitude of oscillation

$$A = \sqrt{x_0^2 + \frac{m}{k}v_0^2}$$

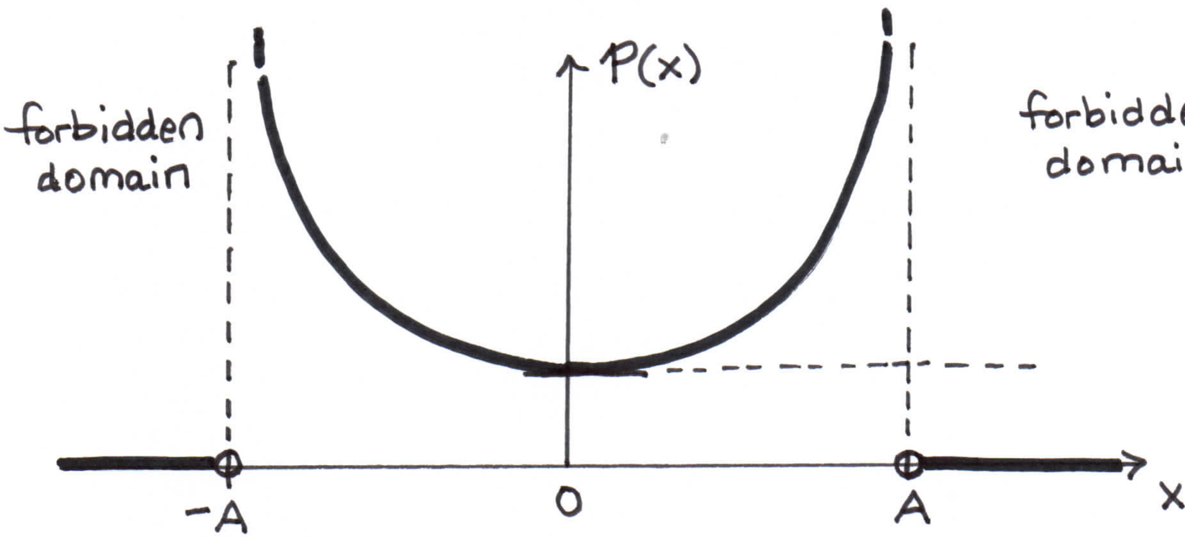
The velocity of the mass at any x-position is given by:

$$v(x) = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

The probability density $P(x)$ of finding the oscillating mass at any position (units of m^{-1}) is inversely proportional to the mass's speed v.

constant of proportionality

$$P(x) = \frac{\alpha}{v} = \frac{\alpha}{\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}}$$



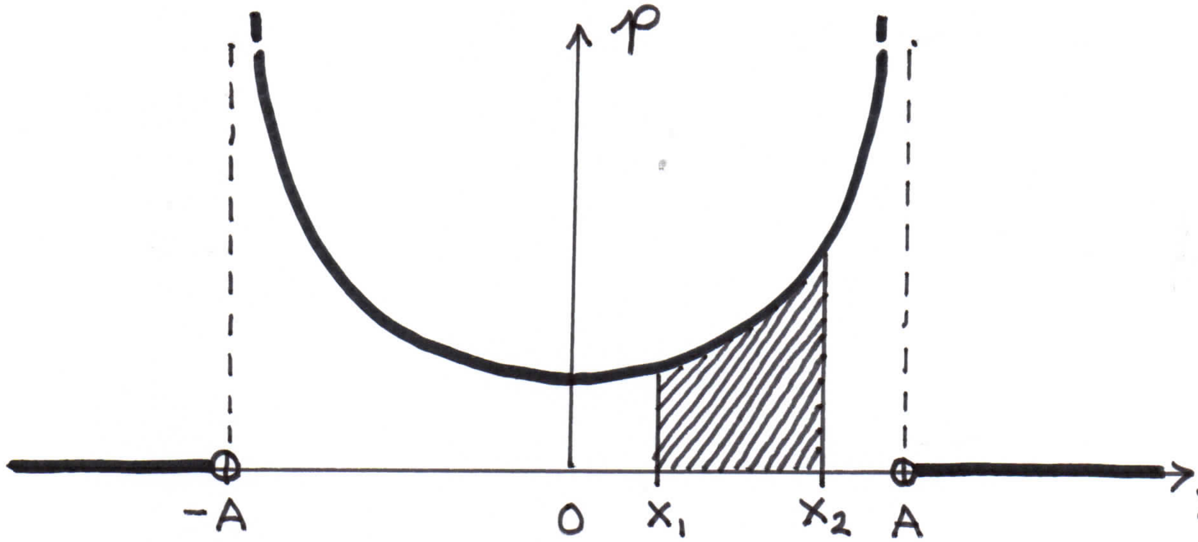
Normalization requires $\int_{-A}^{A} P(x)\,dx = 1$

Find $\alpha = \frac{1}{\pi}\sqrt{\frac{k}{m}}$

So $P(x) = \begin{cases} 0 \text{ for } x < -A \\ \frac{1}{\pi}(A^2 - x^2)^{-1/2} \text{ for } -A < x < A \\ 0 \text{ for } x > A \end{cases}$

The probability of finding the mass between positions $x = x_1$ and $x = x_2$ at any moment is given by

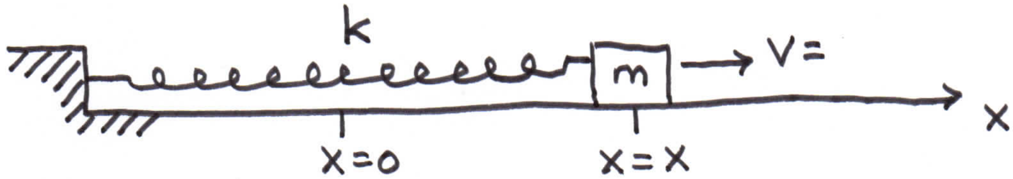
$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} \frac{1}{\pi}(A^2 - x^2)^{-\frac{1}{2}}\, dx$$



Since the total energy of a classical S.H.O. is $\frac{1}{2}kA^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$ and since x_0 and v_0 can be any real numbers of meters or meters/second (forget about limitations on v_0 due to relativity), total energy of a classical S.H.O. can be any non-negative value of joules.

$E=0$ classical E-spectrum for a S.H.O. E

Quantum Mechanical S.H.O.



① Write the Schrödinger Equation for the system using the specific form of the potential energy $U(x) = \frac{1}{2}kx^2$ for a S.H.O.

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}kx^2\psi(x) = E\,\psi(x)$$

② Recast this equation into "standard form" so as to facilitate a solution.

$$\psi''(x) + \frac{2m}{\hbar^2}\left[E - \frac{1}{2}kx^2\right]\psi(x) = 0$$

this is a 2nd order O.D.E. (homogeneous)

③ Solve this equation and find all possible solutions (all possible eigen-functions $\psi_i(x)$ of the operator $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)$ and their correspon-ding eigenvalues E_i).

steady energy states

steady energy values

Solutions to the Schrödinger Equation for a S.H.O.:

stable energy state wave functions (eigenfunctions)

stable state energy values (eigenvalues)

$$\psi_0(x) = C_0 e^{-\frac{m\omega x^2}{2}} \{1\} \qquad E_0 = (0+\tfrac{1}{2})\hbar\omega$$

note: $\omega = \sqrt{\frac{k}{m}}$

$$\psi_1(x) = C_1 e^{-\frac{m\omega x^2}{2}} \left\{2\sqrt{\frac{m\omega}{\hbar}}\, x\right\} \qquad E_1 = (1+\tfrac{1}{2})\hbar\omega$$

$$\psi_2(x) = C_2 e^{-\frac{m\omega x^2}{2}} \left\{4\left(\frac{m\omega}{\hbar}\right) x^2\right\} \qquad E_2 = (2+\tfrac{1}{2}\hbar\omega)$$

$$\psi_3(x) = C_3 e^{-\frac{m\omega x^2}{2}} \left\{8\left(\frac{m\omega}{\hbar}\right)^{3/2} x^3 - 12\sqrt{\frac{m\omega}{\hbar}}\, x\right\} \qquad E_3 = (3+\tfrac{1}{2})\hbar\omega$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\psi_n(x) = C_n e^{-\frac{m\omega x^2}{2}} \left\{H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)\right\} \qquad E_n = (n+\tfrac{1}{2})\hbar\omega$$

where the C_i's are normalization constants given by: $C_n = (2^n n! \sqrt{\pi})^{-1/2}$

and the H_i's are the Hermite polynomials:

$H_0(y) = 1$

$H_1(y) = 2y$

$H_2(y) = 4y^2 - 2$

$H_3(y) = 8y^3 - 12y$

$H_4(y) = 16y^4 - 48y^2 + 12$

$H_5(y) = 32y^5 - 160y^3 + 120y$ ....

Hermite Polynomials

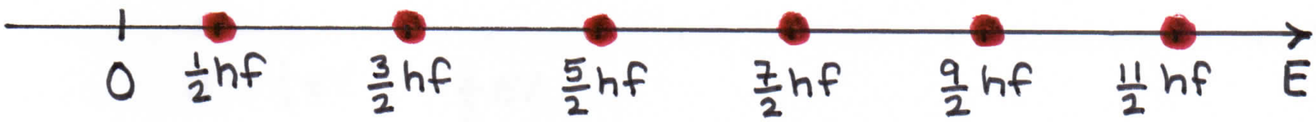
The first 11 Hermite polynomials as used in physics:

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120 \\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x \\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680 \\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x \\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240
\end{aligned}
$$

Thus the allowed energies of a quantum mechanical S.H.O. are given by:

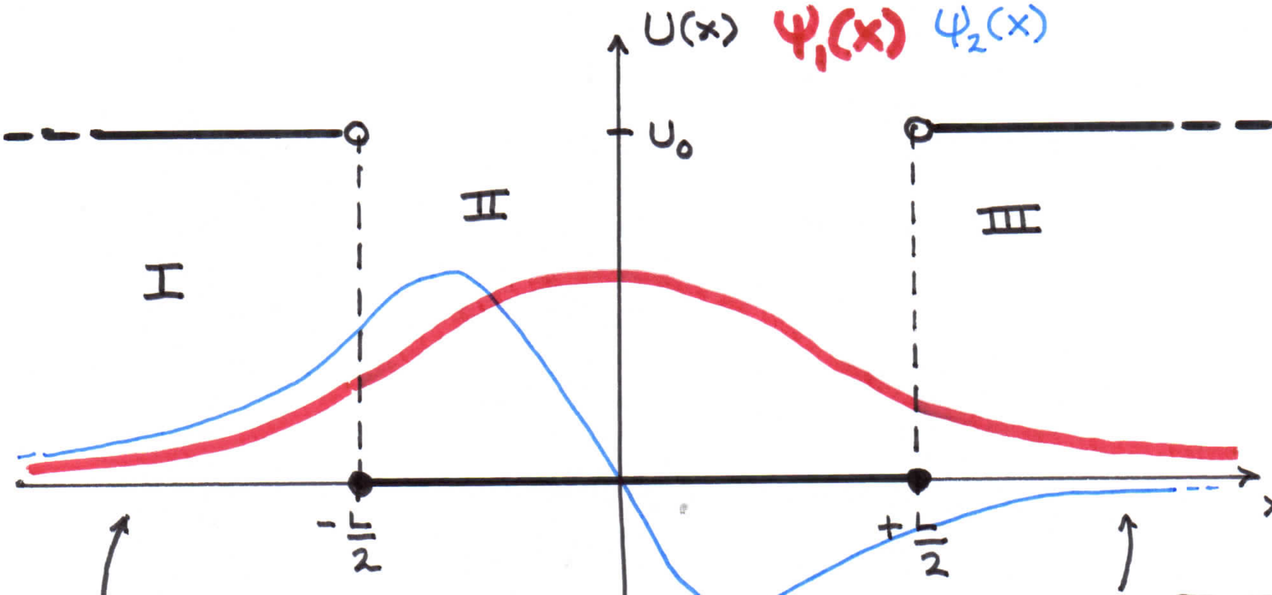
$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \quad \text{or} \quad \left(n+\tfrac{1}{2}\right)hf \qquad n = 0, 1, 2, \ldots\ldots$$

↑ quantum number



energy spectrum of a quantum mechanical S.H.O.

Finite Square Well



$$\psi_I(x) = Ce^{+\sqrt{\frac{2m(U_0-E)}{\hbar^2}}x}$$

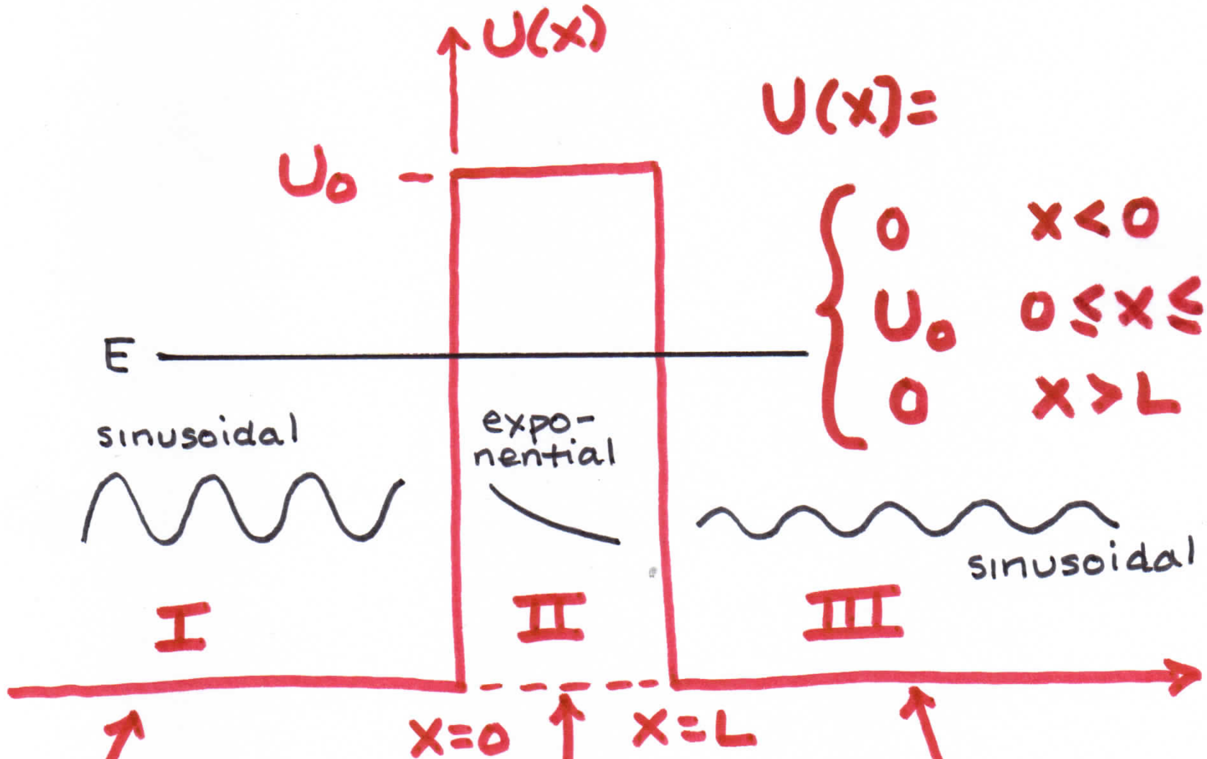
$$\psi_{III} = Ce^{-\sqrt{\frac{2m(U_0-E)}{\hbar^2}}x}$$

$$\psi_{II} = D\cos\left(\sqrt{\frac{2mE}{\hbar^2}}x\right)$$

or

$$D\sin\left(\sqrt{\frac{2mE}{\hbar^2}}x\right)$$

Finite Barrier



$$U(x) = \begin{cases} 0 & x<0 \\ U_0 & 0 \le x \le L \\ 0 & x>L \end{cases}$$

region I:

$$-\frac{\hbar^2}{2m}\psi_I'' = E\psi_I$$

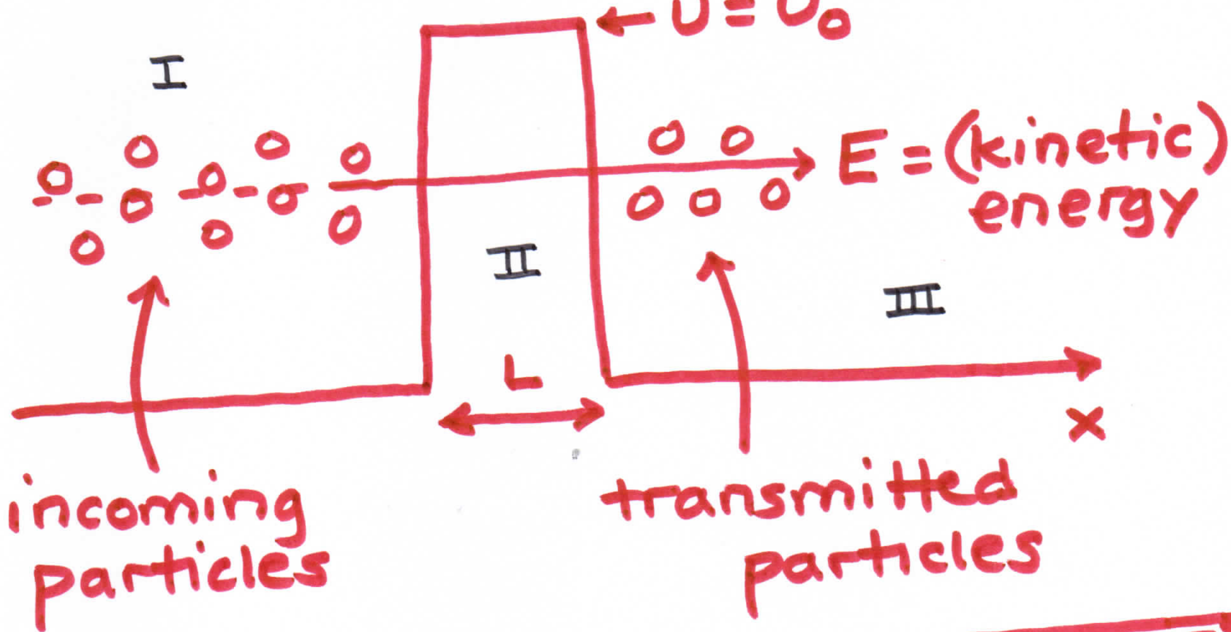
region II:

$$-\frac{\hbar^2}{2m}\psi_{II}'' + U_0\psi_{II} = E\psi_{II}$$

region III:

$$-\frac{\hbar^2}{2m}\psi_{III}'' = E\psi_{III}$$

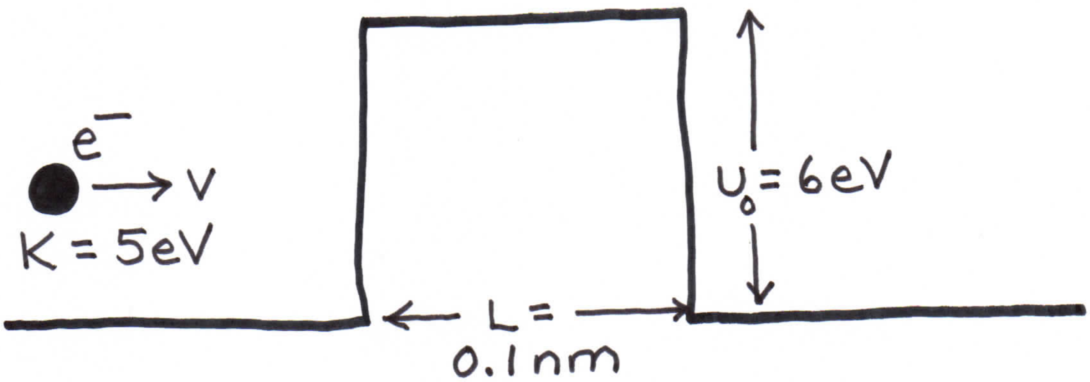
Barrier Transmission



transmission coefficient $\quad T \approx e^{-2L\sqrt{\frac{2m(U_0-E)}{\hbar^2}}}$

$$T = \frac{(\text{amplitude of sinusoidal } \psi_{III})^2}{(\text{amplitude of sinusoidal } \psi_{I})^2}$$

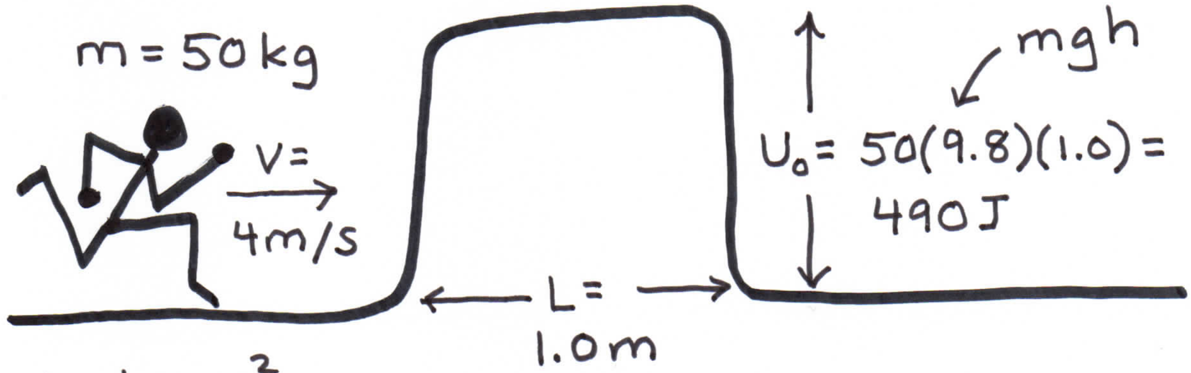
Barrier Penetration Example



$$T = e^{-\frac{2L\sqrt{2m(U_0-E)}}{\hbar}} = e^{-\frac{2L\sqrt{2mc^2(U_0-E)}\,2\pi}{hc}}$$

$$= e^{-\frac{2(.1)\sqrt{2(511{,}000)(6-5)}\,2\pi}{1240}} = \underline{\underline{.359}} \text{ or } \underline{\underline{35.9\%}}$$

Barrier Penetration Example



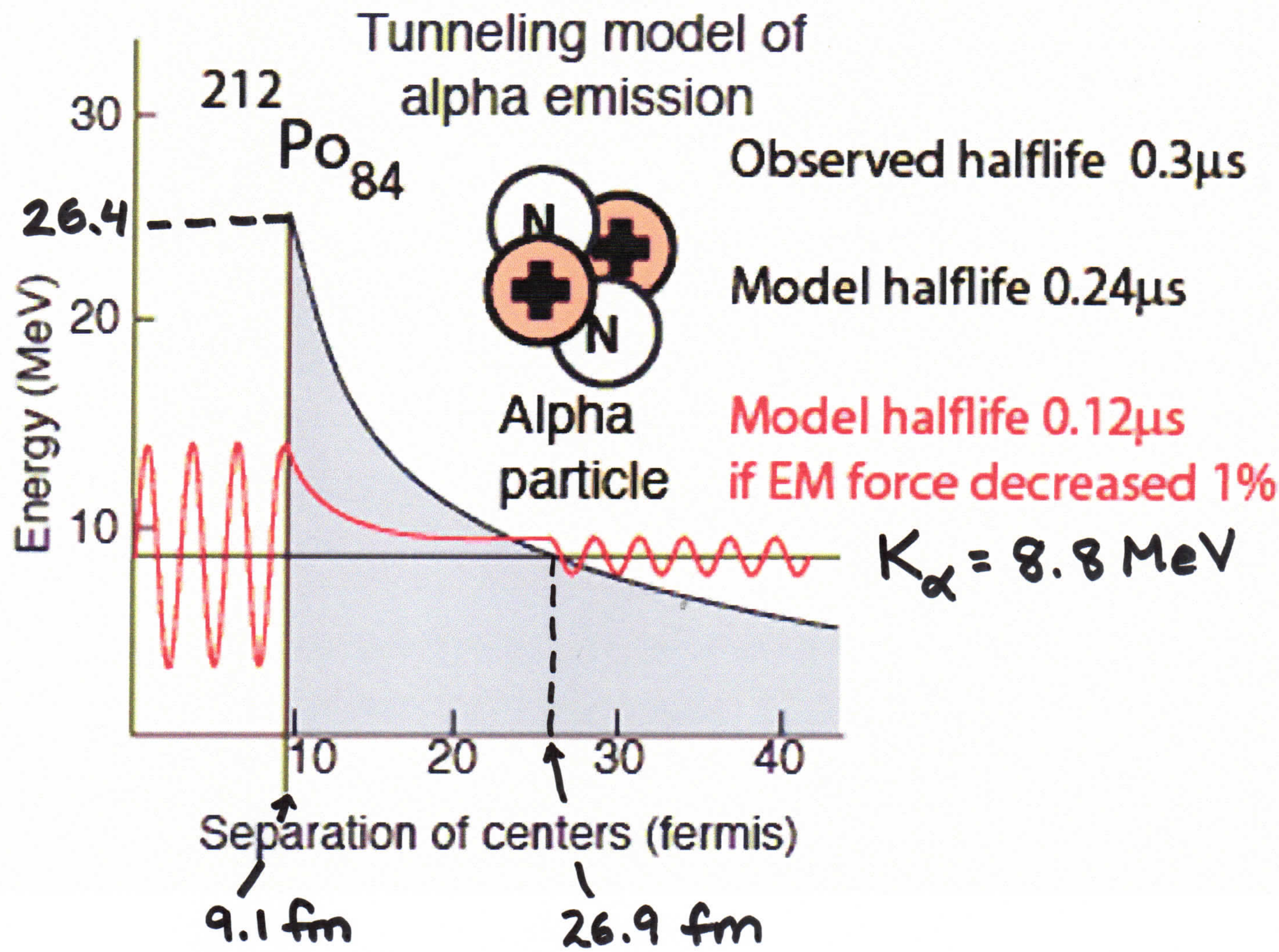
$$K = \frac{1}{2} m v^2 =$$

$$\frac{1}{2}(50)(4)^2 = 400\,\mathrm{J}$$

$$T = e^{-\frac{2L\sqrt{2m(U_0 - E)}}{\hbar}} =$$

$$e^{-\frac{2(1.0)\sqrt{2(50)(490-400)}\,2\pi}{6.63\times10^{-34}}} =$$

$$e^{-1.80\times10^{36}} = \frac{1}{10^{7.8\times10^{35}}}$$


Tunneling model of
alpha emission
212
Po84
26.4
Energy (MeV)
30
20
10
Observed halflife 0.3μs
Model halflife 0.24μs
Model halflife 0.12μs
if EM force decreased 1%
N
N
Alpha
particle
Kα = 8.8 MeV
10
20
30
40
Separation of centers (fermis)
9.1 fm
26.9 fm